Definite integrals & the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

If $f(x)$ is continuous for $c \le x \le d$, and $F(t) = \int_c^t f(x)\,dx$

then $F'(t) = f(t)$ and $\int_a^b f(x)\,dx = F(b) - F(a)$

Compute $\int_0^2 3x^7 + 5x + 2\,dx$, $\int_{-2}^{2} 5e^{3y}\,dy$, $\int_2^5 \frac{1}{3t+2}\,dt$

What is the area enclosed by the x-axis, the curve $y = x^2$ and the lines $x = 2$ and $x = 5$?

Let $g(t) = t(t-1)(t-2)$. What is the area of the region between the x-axis, the curve $y = g(x)$ and the lines $x = 0$ and $x = 2$?

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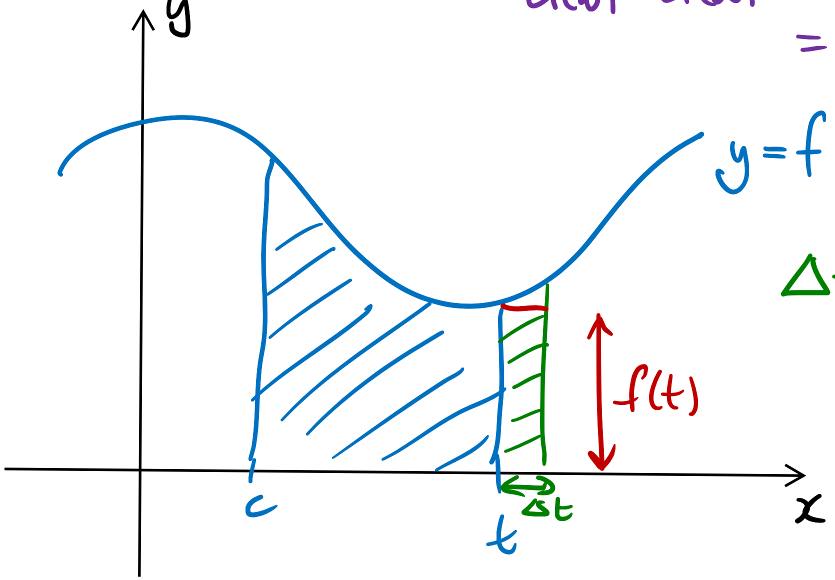
if $G'(t) = f(t)$, then $G(t) = F(t) + C$

$$G(b) - G(a) = (F(b) + C) - (F(a) + C)$$
$$= F(b) - F(a)$$

$$F(c) = \int_c^c f(x)\,dx = 0$$

$$F(b) - F(a) = \int_c^b f(x)\,dx - \int_c^a f(x)\,dx$$
$$= \int_a^b f(x)\,dx.$$



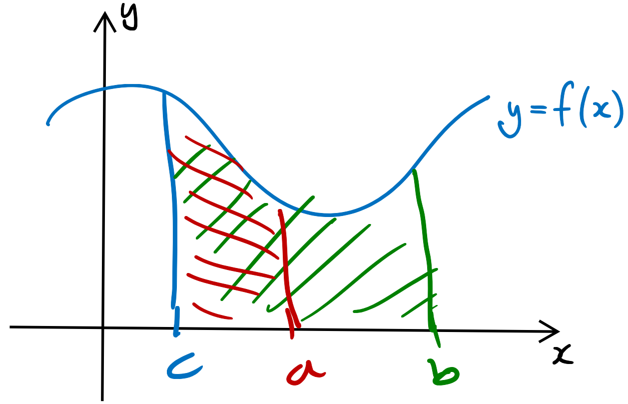
$F(t) = \int_c^t f(x)\,dx$

$$\Delta F \approx f(t) \times \Delta t$$

$$\frac{\Delta F}{\Delta t} \approx f(t)$$

$$\downarrow \Delta t \to 0$$

$$\frac{dF}{dt} = f(t)$$



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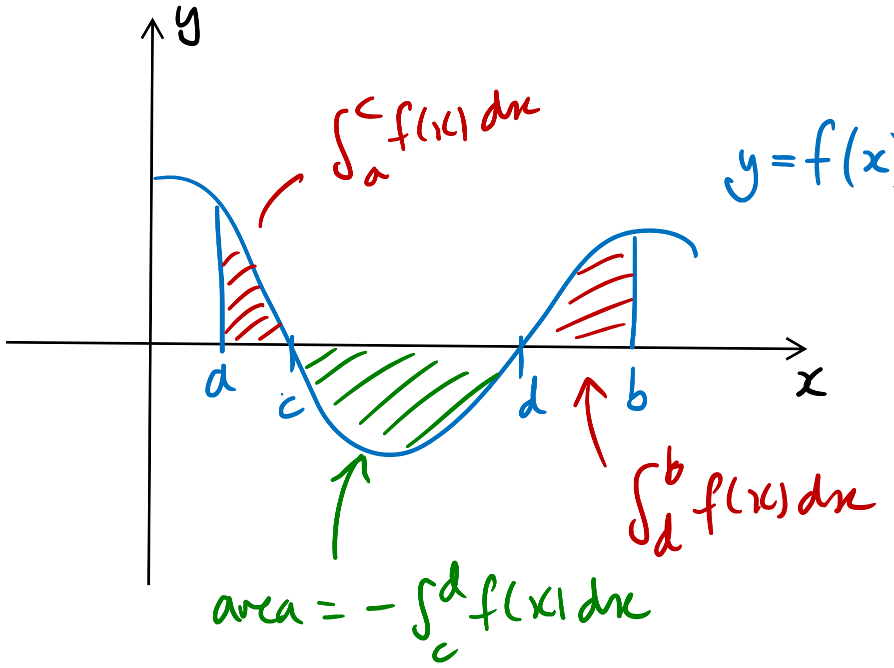

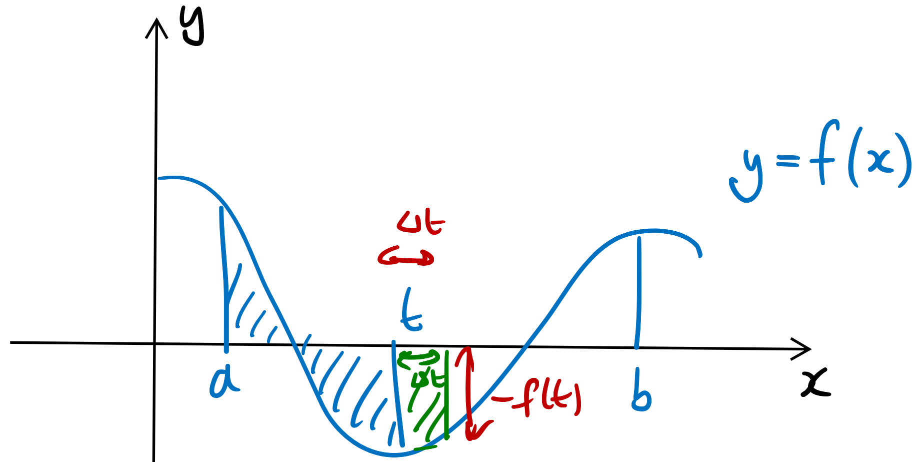

$$F(t) = \int_a^t f(t)\,dt$$

$$\begin{aligned} \Delta F &= -\Delta \text{area} \\ &\approx -(-f(t) \times \Delta t) \\ &= f(t)\,\Delta t \end{aligned}$$

Definite integrals & the Fundamental Theorem of Calculus

Compute $\int_0^2 \underbrace{3x^7 + 5x + 2}\,dx = \int_0^2 \frac{d}{dx}\left(\frac{3}{8}x^8 + \frac{5}{2}x^2 + 2x\right)dx$

$$\left(\begin{array}{lll} \frac{d}{dx}(x^8) = 8x^7 & \frac{d}{dx}(x^2) = 2x & \frac{d}{dx}(x) = 1 \\ \frac{d}{dx}\left(\frac{3}{8}x^8\right) = 3x^7 & \frac{d}{dx}\left(\frac{5}{2}x^2\right) = 5x & \frac{d}{dx}(2x) = 2 \end{array}\right)$$

$$\overset{\text{FTC}}{=} \left[\frac{3}{8}x^8 + \frac{5}{2}x^2 + 2x\right]_0^2$$

$$\frac{2^8}{8} = \frac{2^8}{2^3} = 2^5 = 32$$

$$= \left(\frac{3}{8}2^8 + \frac{5}{2}2^2 + 2\times 2\right) - \left(\frac{3}{8}0^8 + \frac{5}{2}0^2 + 2\times 0\right)$$

$$= (96 + 10 + 4) - 0$$

$$= \underline{\underline{110}}$$

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Compute $\displaystyle\int_{-2}^{2} 5e^{3y}\,dy \overset{\text{FTC}}{=} \left[\frac{5}{3}e^{3y}\right]_{-2}^{2} = \frac{5}{3}e^{6} - \frac{5}{3}e^{-6} = \underline{\underline{\frac{5}{3}\left(e^{6} - e^{-6}\right)}}$

$$\frac{d}{dy}\left(e^{3y}\right) = 3e^{3y}$$

$$\frac{d}{dy}\left(\frac{5}{3}e^{3y}\right) = 5e^{3y}$$

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Compute $\int_2^5 \frac{1}{3t+2}\,dt = \frac{1}{3}\int_2^5 \frac{3}{3t+2}\,dt \overset{FTC}{=} \frac{1}{3}\Big[\ln(3t+2)\Big]_2^5$

$$= \frac{1}{3}\big(\ln(17) - \ln(8)\big)$$

$$= \frac{1}{3}\ln\left(\frac{17}{8}\right)$$

$$\frac{d}{dt}\big(\ln(3t+2)\big) = \frac{3}{3t+2}$$

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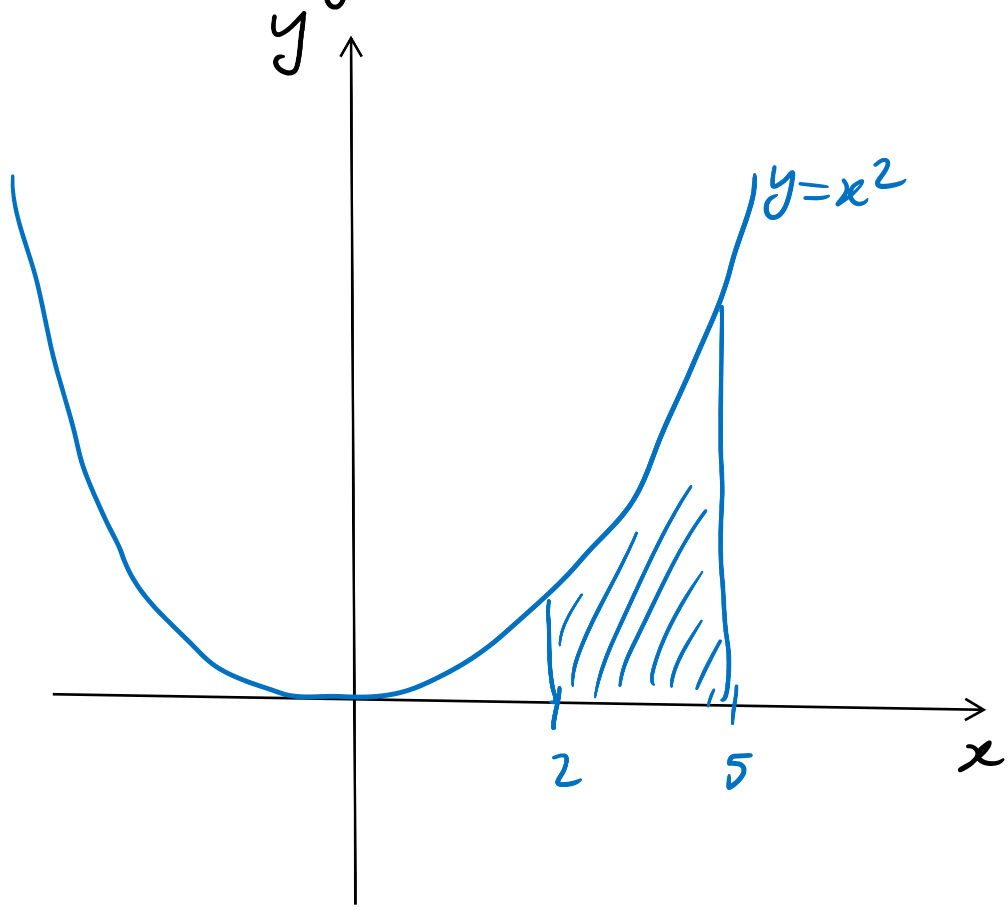
What is the area enclosed by the x-axis, the curve $y = x^2$ and the lines $x = 2$ and $x = 5$?



$$\text{area} = \int_2^5 x^2 \, dx$$

$$= \left[\frac{x^3}{3}\right]_2^5$$

$$= \frac{5^3}{3} - \frac{2^3}{3}$$

$$= \frac{125}{3} - \frac{8}{3}$$

$$= \frac{117}{3}$$

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Let $g(t) = t(t-1)(t-2)$. What is the area of the region between the x-axis, the curve $y = g(x)$ and the lines $x=0$ and $x=2$?

$$\text{area} = \int_0^1 g(t)\,dt - \int_1^2 g(t)\,dt$$

$$g(t) = t^3 - 3t^2 + 2t = \frac{d}{dt}\left(\frac{t^4}{4} - t^3 + t^2\right)$$

$$\int_0^1 g(t)\,dt = \left[\frac{t^4}{4} - t^3 + t^2\right]_0^1$$

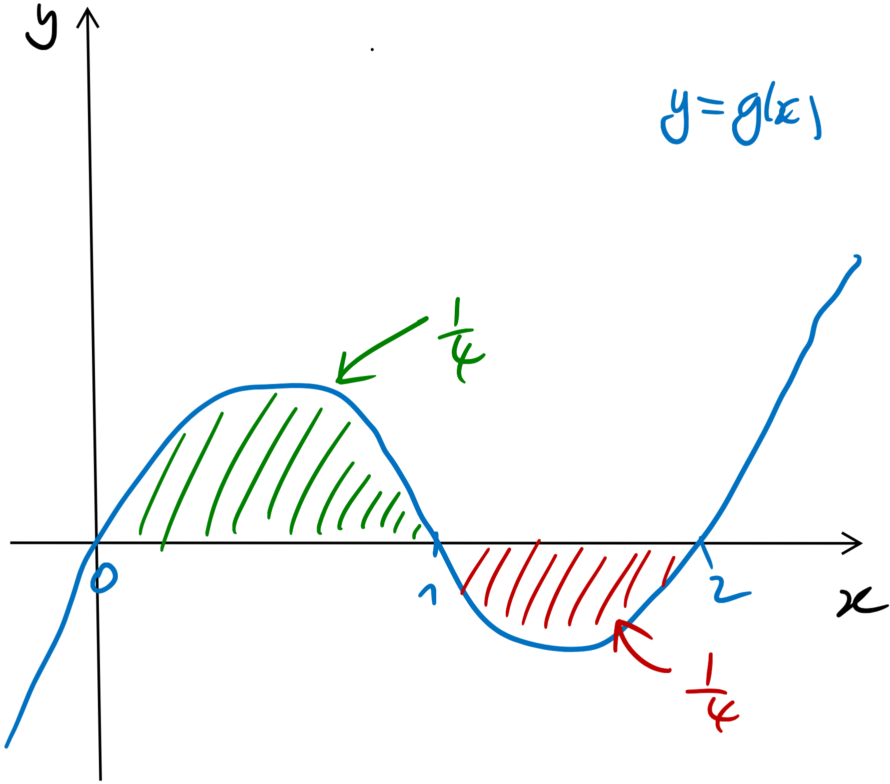
$$= \left(\tfrac{1}{4} - 1 + 1\right) - (0) = \tfrac{1}{4}$$

$$\int_1^2 g(t)\,dt = \left[\frac{t^4}{4} - t^3 + t^2\right]_1^2$$

$$= \left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\tfrac{1}{4} - 1 + 1\right)$$

$$= (4 - 8 + 4) - \left(\tfrac{1}{4}\right) = -\tfrac{1}{4}$$

$$\text{area} = \tfrac{1}{4} - \left(-\tfrac{1}{4}\right) = \underline{\underline{\tfrac{1}{2}}}$$



$$\int_0^2 g(t)\,dt = 0$$