Factorizing Polynomials

Factorize the following:

$$60x^2y^5 + 45x^3y^4 + 15x^4y^3$$

$$24t^3(2t+1) + 36t^2(2t+1)^2$$

$$25p^4 - 4q^2$$

$$x^3 - 2x^2 - 5x + 6$$

$$t^4 - 8t^3 + 18t^2 - 16t + 5$$

Solve $\begin{cases} p = q^2 \\ p^2 - 3p + 5 = q^3 + 8q - 15 \end{cases}$

For which q is $q^4 + q^2 < 20$

Factorizing Polynomials

Factorize the following:

$$60x^2y^5 + 45x^3y^4 + 15x^4y^3 = (15x^2y^3)(4y^2) + (15x^2y^3)(3xy) + (15x^2y^3)x^2$$

$15\times4\ x^2y^3y^2$

$15\times3\ x^2xy^3y$

$15\times1\ x^2x^2y^3$

$$= 15x^2y^3(4y^2 + 3xy + x^2)$$

$$24t^3(2t+1) + 36t^2(2t+1)^2 = [12t^2(2t+1)][2t + 3(2t+1)]$$

$2\times12\ t^2t\ (2t+1)$

$3\times12\ t^2(2t+1)(2t+1)$

$$= 12t^2(2t+1)(2t + 6t + 3)$$

$$= 12t^2(2t+1)(8t+3)$$

Factorizing Polynomials

Factorize the following:

$$25p^4 - 4q^2 = 5^2(p^2)^2 - 2^2q^2$$

$$= (\underbrace{5p^2}_{x})^2 - (\underbrace{2q}_{y})^2$$

$$= (5p^2+2q)(5p^2-2q)$$

difference of 2 squares

$$(x+y)(x-y) = x^2 - xy + xy - y^2$$

$$= x^2 - y^2$$

Factorizing Polynomials

The Factor Theorem

If $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$ is a polynomial then $f(c) = 0$ if and only if $x - c$ divides $f(x)$.

$$f(x) = x^2 + 3x - 10 = (x-2)(x+5)$$

$f(0) = 0^2 + 3\times 0 - 10 = -10 \neq 0$ $x - 0$ is not a factor of $f(x)$

$f(1) = 1^2 + 3\times 1 - 10 = 1 + 3 - 10 = -6 \neq 0$ so $x-1$ isn't factor

$f(2) = 2^2 + 3\times 2 - 10 = 4 + 6 - 10 = 0$ so $\underline{x-2 \text{ is a factor}}$

$f(-5) = (-5)^2 + 3(-5) - 10 = 25 - 15 - 10 = 0$ so $x - (-5) = \underline{x+5}$ is a factor.

$$x^2 + 3x - 10 = (x-2)(x+5)$$

Factorizing Polynomials

Factorize $x^3 - 2x^2 - 5x + 6 = f(x)$

1, 2, 3, 6

$f(1) = 1^3 - 2\times1^2 - 5\times1 + 6 = 1 - 2 - 5 + 6 = 0$ $(x-1)$ is a factor

$$x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 + bx - 6)$$

$-2x^2 = bx^2 - x^2$

$-2x^2 = (b-1)x^2$

$-2 = b - 1$

$b = -1$

$$f(x) = x^3 - 2x^2 - 5x + 6 = (x-1)(x^2 - x - 6)$$

$g(x) = x^2 - x - 6$,

$g(1) = 1 - 1 - 6 \neq 0$

$g(-1) = 1 + 1 - 6 \neq 0$

$g(2) = 4 - 2 - 6 \neq 0$

$g(-2) = 4 + 2 - 6 = 0$ So $x - (-2) = x + 2$ is a factor of g

$$x^2 - x - 6 = (x+2)(x-3)$$

$$\underline{\underline{f(x) = (x-1)(x+2)(x-3)}}$$

Factorizing Polynomials

Factorize $t^4 - 8t^3 + 18t^2 - 16t + 5 = f(t)$

$f(1) = 1 - 8 + 18 - 16 + 5 = 0$ so $t - 1$ is a factor

$f(-1) = 1 + 8 + 18 + 16 + 5 \neq 0$

$$
\begin{aligned}
f(5) &= 5^4 - 8 \times 5^3 + 18 \times 5^2 - 16 \times 5 + 5 \\
&= 25 \times 25 - 40 \times 25 + 18 \times 25 - 15 \times 5 \\
&= 25(25 - 40 + 18) - 15 \times 5 \\
&= 25 \times 3 - 15 \times 5 \\
&= 3 \times 5^2 - 3 \times 5^2 = 0
\end{aligned}
$$

so $t - 5$ is a factor

$$
\begin{aligned}
t^4 - 8t^3 + 18t^2 - \underline{16t} + 5 &= (t-1)(t-5)(at^2 + bt + c) \\
&= (t^2 - \underline{6t} + 5)(t^2 + 6t + \underline{1}) \\
&= (t^2 - 6t + 5)(t^2 - 2t + 1) \\
&= (t-1)(t-5)(t-1)^2 \\
&= \underline{\underline{(t-1)^3(t-5)}}
\end{aligned}
$$

$-16t = -6t + 5bt$

$-16 = -6 + 5b$

$-10 = 5b$

$b = -2$

Factorizing Polynomials

Solve $\begin{cases} p = q^2 & \text{--- (1)} \\ p^2 - 3p + 5 = q^3 + 8q - 15 & \text{--- (2)} \end{cases}$

Sub $p = q^2$ into (2): $(q^2)^2 - 3(q^2) + 5 = q^3 + 8q - 15$

$$q^4 - 3q^2 + 5 = q^3 + 8q - 15$$

$$f(q) = q^4 - q^3 - 3q^2 - 8q + 20 = 0$$

$f(1) = 1 - 1 - 3 - 8 + 20 \neq 0$

$f(-1) = 1 + 1 - 3 + 8 + 20 \neq 0$

$f(2) = \cancel{16} - 8 - 12 - \cancel{16} + 20 = 0$ so $q-2$ is a factor of f

$$q^4 - q^3 - 3q^2 - 8q + 20 = (q-2)(q^3 + aq^2 + bq - 10)$$

$-8q = -10q - 2bq$

$-8 = -10 - 2b$

$2 = -2b$

$b = -1$

$-3q^2 = -2aq^2 - q^2$

$-3 = -2a - 1$

$-2 = -2a$

$a = 1$

$$= (q-2)(q^3 + q^2 - q - 10)$$

$$f(q) = (q-2)(q-2)(\underbrace{q^2 + 3q + 5}_{\text{discriminant}})$$

$\Delta = 3^2 - 4 \times 5 = 9 - 20 < 0$

$f(q) = 0 \iff q - 2 = 0 \iff \underline{\underline{q = 2 \ \& \ p = 4}}$

Factorizing Polynomials

For which q is $q^4+q^2<20$

$\iff q^4+q^2-20<0$

Method ① $f(q)=q^4+q^2-20$

$f(\pm 1)=1+1-20\neq 0$

$f(\pm 2)=16+4-20=0$

so $q-2$ & $q-(-2)=q+2$ are factors

$q^4+q^2-20=(q-2)(q+2)(q^2+bq+5)$

$0q=10q-10q-4bq$

$0=-4b, \quad b=0$

$q^4+q^2-20=(q-2)(q+2)(q^2+5)$

Method ② Let $u=q^2$, then

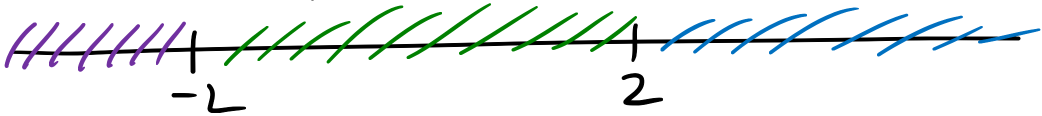
$q^4+q^2-20=u^2+u-20=(u-4)(u+5)$

$=(q^2-4)(q^2+5)=(q-2)(q+2)(q^2+5)$

When is $f(q)<0$?

$\underbrace{(q-2)}_{\substack{>0 \text{ if } q>2\\ <0 \text{ if } q<2}}\underbrace{(q+2)}_{\substack{>0 \text{ if } q>-2\\ <0 \text{ if } q<-2}}\underbrace{(q^2+5)}_{>0}<0$

$f(2)=0, \ f(-2)=0$



$q<-2$: $q-2<0$, $q+2<0$, $q^2+5>0$ $\quad f(q)>0$

$\boxed{-2<q<2}$: $q-2<0$, $q+2>0$, $q^2+5>0$ $\quad f(q)<0$

$q>2$: $q-2>0$, $q+2>0$, $q^2+5>0$ $\quad f(q)>0$