Interactive Economics Facts and Formulae

This document summarises some main mathematical ideas that you will probably see in the first year of any economics degree course. The hot links allow you to select questions, each randomised and with full feedback so you can 'get your hands dirty' and reinforce your understanding. You are encouraged to make good use of these links and to retain this document as a handy summary for revision. You/your teacher is free to edit it as required. You will find questions on additional topics in economics, as well as most of the underlying mathematical techniques, in the maths e.g. database.

Indices and Mathematics of Finance

The share price index, P_y, in year y with a base year b is $P_y/P_b \times 100\%$.

Index numbers like RPI, CPI, use *scale factors*: for example suppose the turnover of a company in 2014 was £1.92m and the RPI for 2014 was 148.21, but was 110.18 in 2012. Then the turnover for 2014 in 2012 prices is £1.92m×(110.18/148.21) = £1.43m approx.

Percentage changes can also be thought of as *scale factors*: for example if a price P rises by 30% and then drops by 40% the new price is $P(1+30/100)(1-40/100)$ = P(0.78) i.e. a drop of (1-.78)×100 = 22%. Similarly if the value P appreciates by s% per year, the value after n years is $P(1+s/100)^n$. If $s < 0$ then the same formula holds for depreciation. Inflation works in a similar way: suppose we have inflation of 5% followed by 8%, then at the end of the two years we see that a good that cost P_0 will cost $P_2=P_0(1.05)(1.08)$ in two years. To go forwards in time we multiply by the scale factor and to go backwards we divide e.g. $P_0=P_2/[(1.05)(1.08)]$.

Suppose we have m goods e.g. different fuels (gas, oil etc) being sold in different quantities in each year n at $P_{g,n}$ in different quantities $Q_{g,n}$. Then with respect to the base year 0, the Lasperye index is $\sum_{g=1}^{m} \frac{P_{g,n}Q_{g,0}}{P_{g,0}Q_{g,0}}$ and the the Paasche index is $\sum_{g=1}^{m} \frac{P_{g,n}Q_{g,n}}{P_{g,0}Q_{g,0}}$

The Herfindal index H measures the competition in a market with n companies, each having a different market share of S_i%. Then $H = \sum_{i=1}^{n} {S_i}^2$. For many small companies H is small but rises to 10,000 for a monopoly.

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Simple Interest the value of an investment $S = P \times t \times \left(1 + \frac{r}{100}\right)$ where *P* is the principal, *t* is the number of years invested and *r* is the nominal annual rate of interest in percent.

For Discrete Compound Interest we have $S = P \times \left(1 + \frac{r}{100}\right)^{t}$.

Compounding *m* times per year gives $S = P \times \left(1 + \frac{r/m}{100}\right)^{mt}$.
Taking the limit as $m \to \infty$ gives $S = Pe^{rt/100}$ for continuous compounding.
The annual percentage rate (APR) is related to the scale factor for one year in each case. So for a nominal rate of 20% compounded continuously $APR = \left(e^{20/100} - 1\right) \times 100 \approx 22.14\%$.

The present value (PV) of a future income *S* is given by rearranging the above discrete or continuous compound interest formulae to give *P*. The net present value (NPV) is the cost – PV, hence:

$NPV = C - S \times \left(1 + \frac{r}{100}\right)^{-n}$ for the discrete case and

$NPV = C - S \times \left(1 + \frac{r/m}{100}\right)^{-mt}$ for the continuous case.

If NPV > 0 then the project is recommended, if NPV < 0 it is not recommended. You cannot compare two projects on the basis of their NPV's unless *S* is the same; for dissimilar values of *S*, you should evaluate the internal rate of returns instead; this requires finding *r* in the formulae for compound interest above, given the values of all the other variables. For example continuous compounding gives $r = \frac{100}{t} \ln\left(\frac{S}{P}\right)$. If IRR > the market rate, the project is recommended. If not, then it is not recommended.

The value of a sinking fund at the end of *n* years with investment *P* made at the start of each year is a geometric progression (GP); for discrete compounding:

$$S = P \times \left(1 + \frac{r}{100}\right)\left\{1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^{2} + \cdots + \left(1 + \frac{r}{100}\right)^{n-1}\right\}$$

whilst for continuous compounding $S = Pe^{r/100}\left\{1 + e^{r/100} + e^{2r/100} + \cdots + e^{(n-1)r/100}\right\}$.
Both can be summed using the standard formula for a GP.

The present value (PV) of an annuity of S paid at the end of every year is:

$$PV = S \times \left(1+\frac{r}{100}\right)^{-1}\left\{1+\left(1+\frac{r}{100}\right)^{-1}+\left(1+\frac{r}{100}\right)^{-2}+\cdots+\left(1+\frac{r}{100}\right)^{1-n}\right\}$$ (discrete case) or

$$PV = Se^{-r/100}\left\{1+e^{-r/100}+e^{-2r/100}+\cdots+e^{(1-n)r/100}\right\}$$

(continuous case)

which again can be summed using the standard formula. If these are summed to infinity, you get the present value of a perpetuity.

Topics in Microeconomics

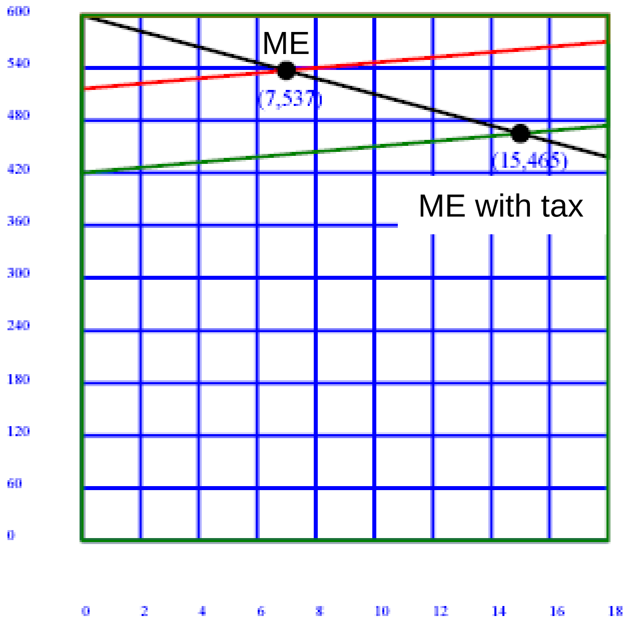
In a perfectly competitive market, an individual company cannot influence the price P, but only the quantity Q of the good that it produces. Thus P (the dependent variable) is a function of Q (the independent variable), i.e. $P(Q)$. Both are endogenous variables (within the model). The simplest model of the DEMAND curve is of the form

$P_d = a - bQ$

whilst the SUPPLY curve is $P_s = c+dQ$,

where the parameters a,b,c,d are positive constants. Such models should not be taken too literally – they are really approximations to the supply and demand curves and are valid only near the market equilibrium (see below). The demand curve can also depend on other (exogenous) variables such as average consumer income Y and price of an alternative good P_a; for example $P = 20 - 3Q + eY + f P_a$; for a positive/negative parameter e, the good is normal(or superior)/inferior and for positive/negative f, the alternative good is substitutable/complementary.

Market equilibrium (ME) occurs when $P_d = P_s$, solved using algebra or by finding the intersection of the supply and demand curves graphically. If a tax (T)/subsidy (S) is imposed/given per good, the demand curve is unaltered but the supply curve changes to $P_s - T = c+dQ$, or $P_s + S = c+dQ$, moving this curve UP/DOWN and the equilibrium point along the demand curve to the left/right. The example shows demand and supply curves given by $P = 600 - 9Q$ & $P = 420 + 3Q$ and T = 96 giving a pre-tax/post-tax ME at (15,465) and (7,537) respectively. Q falls (by 8). We see that the customer pays £537 - £465 = £72 more so the remaining tax of £24 is actually paid by the company. For subsidies, *Q* increases and the customer and company share the subsidy in a similar manner.



Total cost can be modelled by *TC* = *FC*+*VC*, where *FC* are fixed (or autonomous) costs and *VC* are variable costs, e.g. *TC* = 100+5*Q,* where each good costs 5 additional currency units to produce. Average cost $AC = \frac{TC}{Q}$ .Total revenue $TR = P \times Q$. If *P* is given by a linear demand curve, as above, this is a quadratic quantity in *Q*. Average revenue $= \frac{TR}{Q}$. Profit, $\Pi = TR - TC$ which can be set equal to zero for break-even points or optimised (e.g. using differentiation).

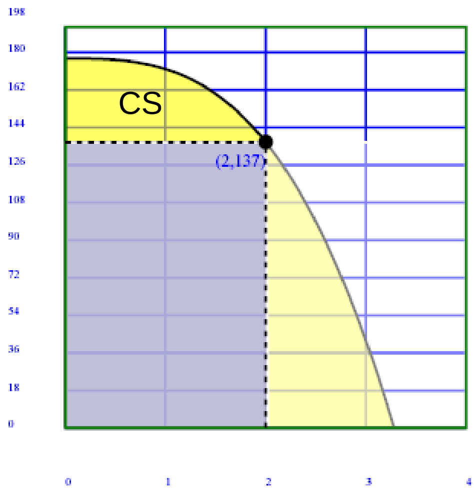
Applications of differentiation

Marginal quantities are derivatives and all work in a similar way. Thus the marginal revenue is $MR = \frac{d\,TR}{dQ}$ so that a finite change ΔQ gives a corresponding change $\Delta TR \approx MR \times \Delta Q$. Thus *MR* is the extra revenue due to an increase in a large production quantity by 1. Similarly for the marginal cost *MC*. At optimum profit $\frac{d\Pi}{dQ} = 0$ so then *MR* = *MC*.

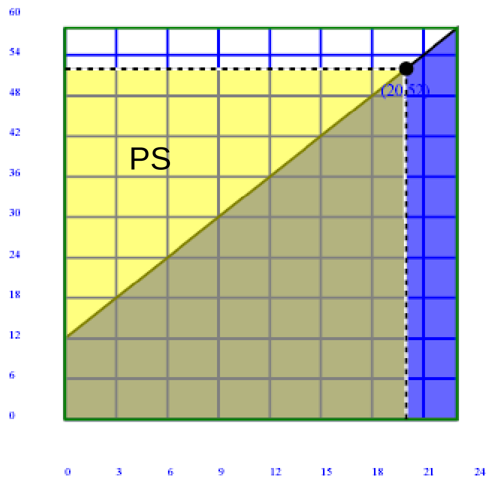
If a firm produces a small increase ΔQ in production, the decreasing demand curve will mean it will sell as a lower price $P - \Delta P$. The price elasticity gives a way to determine what will happen to *TR*. Thus $\Delta TR = (P + \Delta P)(Q + \Delta Q) - PQ \approx \Delta P \times Q + \Delta Q \times P$. If there is no change, $\Delta TR = 0$ and so $E = -\frac{P}{Q}\frac{\Delta Q}{\Delta P} = 1$. This is known as the arc elasticity but is ambiguous since one has to agree to use *P* and *Q* before, after or in the middle of the change. Taking the limit as $\Delta Q \to 0$ removes this ambiguity and gives the (point) elasticity as $E = -\frac{P}{Q}\frac{\mathrm{d}Q}{\mathrm{d}P} = -\frac{P}{Q}\frac{1}{\frac{\mathrm{d}Q}{\mathrm{d}P}}$. If *E* > 1 the demand is elastic and increasing *Q* increases *TR*. If *E* < 1 the demand is inelastic and increasing *Q* decreases *TR*.

Applications of Integration

At a fixed point (Q_0,P_0) on the demand curve P = f(Q) the consumer surplus is $CS = \int_0^{Q_0} f(Q)dQ - P_0\, Q_0$. Graphically it is the area marked in the example (here $P = 177 - 5Q^3$giving CS = 60 when Q=2).



At a fixed point (Q_0,P_0) on the supply curve P = g(Q) the producer surplus is $PS = P_0 Q_0 - \int_0^{Q_0} g(Q)dQ$. Graphically it is the area marked in the example (here $P = 12 + 2Q$
giving PS = 400 when Q=20).



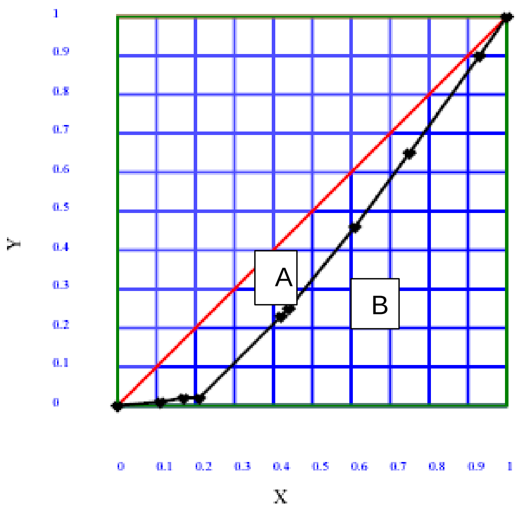
The income distribution of a country can be measured by the [Gini coefficient](). Suppose:

Cumulative percent of households	11	17	21	42	44	61	75	93	100
Cumulative share of national income	1	2	2	23	25	46	65	90	100

It is convenient to scale this to 1 and index the points so that:

k	0	1	2	3	4	5	6	7	8	9
X_k	0	0.11	0.17	0.21	0.42	0.44	0.61	0.75	0.93	1
Y_k	0	0.01	0.02	0.02	0.23	0.25	0.46	0.65	0.9	1

Plotting this gives:



The black curve is the Lorenz curve whilst the red line is that of equality of income. Gini coefficient $G = \frac{A}{A+B} = 2A$. Area *B* can be calculated by the trapezoidal rule so that $G = 1 - \sum_{k=1}^{N}(X_i - X_{i-1})(Y_i + Y_{i-1})$ (=0.2453 in this case). For continuous data the Lorenz curve can be described by a function *L*(*x*) so then $G = 1 - 2\int_0^1 L(x)dx$. Suitable functions for *L*(*x*) must have increasing slope and pass through (0,0) and (1,1); candidates are x^p with *p* > 1, *x* exp(*x*)/e, *x* m^x/*m* with *m* > 1 etc. The Gini coefficient for wealth distribution is defined in the same way, but the Lorenz curve is now that for cumulative share of wealth, not income.

Functions of several variables

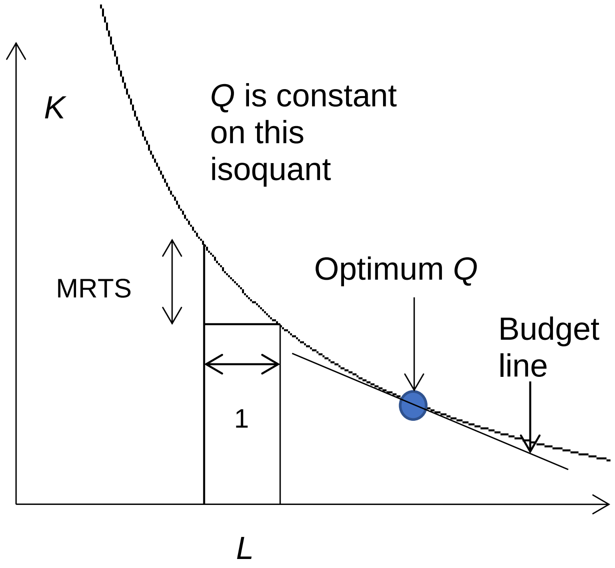
At equilibrium, a linear two-sector model of National Income Determination has $= C + I$, $C = aY + b \; where \;\; 0 < a < 1, b > 0$ and $I = I^* where \;\; I^* > 0$ where *Y* is the (mean) flow of money to households, *C* is household consumption (made up of autonomous and income-dependent consumption) and I^* is a fixed rate of investment into firms. $Y = \frac{b+I^*}{1-a}$ which is a function of three exogenous variables $a, b \; and \; I^*$. The effect of varying these parameters by small amounts can be deduced from the general relationship $\Delta Y \approx \frac{\partial Y}{\partial x}\Delta x$ where *x* is any of the parameters. Thus we can define the *marginal propensity to consume multiplier* as $\frac{\partial Y}{\partial a} = \frac{b+I^*}{(1-a)^2}$, the *autonomous consumption multiplier* as $\frac{\partial Y}{\partial b} = \frac{1}{1-a}$ and the *investment multiplier* as $\frac{\partial Y}{\partial I^*} = \frac{1}{1-a}$.

The model can be extended to include the effect of Government expenditure G and tax T, giving a three-sector model (firms, households and government). Thus, for a linear model we have: $Y = C + I + G$, $C = aY_d + b$, $Y_d = Y - T$, $T = tY + T^* \; where \; 0 < t < 1 \; and \; T^* \geq 0$, $I = I^*$ and $G = G^* \; where \; G^* > 0$. In the above, Y_d is the household *disposable income*. By successive substitutions $Y = \frac{-aT^*+b+I^*+G^*}{1-a+at}$. To examine the effect of government expenditure (G^*) and autonomous taxation (T^*) we define the *government expenditure multiplier* as $\frac{\partial Y}{\partial G^*} = \frac{1}{1-a-at}$ and the *autonomous taxation multiplier* as $\frac{\partial Y}{\partial T^*} = \frac{-a}{1-a-at}$. For a *balanced budget* ,where the government finances a rise in expenditure by a rise in autonomous taxation (i.e. $\Delta G^* = \Delta T^*$), then $\Delta Y \approx \frac{\partial Y}{\partial G^*}\Delta G^* + \frac{\partial Y}{\partial T^*}\Delta T^* = \left(\frac{1-a}{1-a+at}\right)\Delta G^*$ where the term in brackets is the *balanced budget multiplier* which is between 0 and 1.

Production functions typically depend on at least the labour employed (*L*) and capital spend (*K*), so that *Q*(*L*,*K*). If $Q(\lambda L, \lambda K) = \lambda^n Q(L, K)$ the production function is homogeneous; if *n* > 1, =1 or is <1 then *Q*(*L*,*K*) has increasing, constant or decreasing returns to scale. For example with *n* = 3, doubling *L* and *K* would increase *Q* by a factor of 8. An example is the Cobb-Douglas production function $Q(L, K) = A\, L^\alpha K^\beta$, where parameters A, α, β are constants specific to the production line. This is homogeneous with $n = \alpha + \beta$.

The effect on *Q* of small increases in *L* or *K* are quantified by the marginal products of labour and capital: $MP_L = \frac{\partial Q}{\partial L}$ and $MP_K = \frac{\partial Q}{\partial K}$ respecitively. One can also define further partial elasticities & normal/inferior & substitutable/complementary; for example consider the inverse demand curve *Q*(*P*,*Y*,*P*$_a$), for example $Q = 100 - 2P + 0.1Y + P_a$. Here if $Q \uparrow \text{as } Pa \uparrow$ so the alternative good is substitutable and the cross-price elasticity of demand $XED = \frac{P_a}{Q} \times \frac{\partial Q}{\partial P_a}$ is positive; if the opposite happens then the alternative good is complementary. Also $Q \uparrow \text{as } Y \uparrow$ in this example so the good is superior. Similarly, if the income elasticity of demand $YED = \frac{Y}{Q} \times \frac{\partial Q}{\partial Y}$ is positive/negative, the good is superior/inferior.

For realistic production functions, increasing *L* or *K* increases *Q*. In order to preserve the value of *Q* (i.e. stay on the *isoquant)* when decreasing *L* by one unit, we will need to increase *K* by the *marginal rate of technical substitution* or *MRTS*, see diagram. Since $dQ = \frac{\partial Q}{\partial L}dL + \frac{\partial Q}{\partial K}dK = 0$ on this isoquant and *dL* = -1 we have $MRTS = \frac{\frac{\partial Q}{\partial L}}{\frac{\partial Q}{\partial K}} = \frac{MP_L}{MP_K}$. An exactly analogous situation occurs with a *Utility* function of the form $U(x_1, x_2)$ where x_1 and x_2 denote the numbers of goods 1 and 2 and x_1, x_2 replace *L*,*K*. In this case we have marginal utilities and the *marginal rate of commodity substitution* or *MRCS* given by $MRCS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}}$. Thus an increase of *MRCS* in x_2 is needed to remain on the *indifference curve* if x_1 decreases by 1.



Economic quantities of interest can depend on several variables, for example profit might depend on the quantities produced of two goods. In order to optimise an *objective* function of two variables $Z = f(x, y)$ with no constraints we find the partial derivatives set these equal to 0 i.e. $f_x = f_y = 0$. This gives the critical point(s) (CP). To determine their nature we evaluate the Hessian of second derivatives at the CP(s):

$$H = f_{xx}f_{yy} - [f_{xy}]^2$$

If *H* < 0 the CP is a saddle point. If *H* > 0 then f_{xx} and f_{yy} must have the same sign so we need only check one of them. If $f_{xx} > 0$ the CP is a (local) minimum, if $f_{xx} < 0$ it is a (local) maximum. Generally, one has constraints, for example production limits or budget constraints. In its simplest form on wants to optimise $Z = f(x, y)$ subject to a constraint equation $\varphi(x, y) = M$ where *M* is the 'budget'. Using the constraint equation it is sometimes possible to solve for one variable, *y* say, in terms of the other (*x*) and substitute for *y* into the objective function making it a function of *x* alone, and hence optimising using the usual methods of calculus

of a single variable (and determining the nature of the CP). Once known *y* and the value of *Z* can be found.
An alternative method is to find the Lagrangian:
$g(x,y,\lambda) = f(x,y) + \lambda[M - \varphi(x,y)]$. One then solves: $\frac{\partial g}{\partial x} = 0$, $\frac{\partial g}{\partial y} = 0$, $\frac{\partial g}{\partial \lambda} = 0$ to give the (x,y) coordinates of the CP(s). If the equations are linear, this is possible using simultaneous equation/matrix methods; otherwise it can be hard or impossible analytically. The value of λ gives an estimate of the change ΔZ of a small change in budget of ΔM i.e. $\Delta Z \approx \lambda \times \Delta M$.
Finally suppose *Z*(*x*,*y*) is a production function *Q*(*L*,*K*) and one has a budget constraint $P_K.K + P_L.L = M$ where P_K and P_L are the prices of capital and labour respectively. Then $K = -\frac{P_L}{P_K}.L + \frac{M}{P_K}$ so the slope of the budget line is $-\frac{P_L}{P_K}$, see diagram. So at optimum, $\frac{MP_L}{MP_K} = \frac{P_L}{P_K}$. A similar relation holds for a Utility indifference curve.

Ordinary differential and difference equations

Dynamic continuous models of National income and market equilibrium can result in *ordinary differential equations* of the form: $\frac{dy}{dt} = \alpha y + \beta$. This linear equation has its *particular integral* of the form *y* = *D* and substitution gives $D = -\frac{\beta}{\alpha}$. The *complementary function* is $y = Ae^{at}$ where *A* is a constant to be determined by a boundary condition, often the initial condition *y*(0). The solution is then $y = Ae^{\alpha t} - \beta/\alpha$. If $a < 0$ the solution is stable and $\rightarrow -\frac{\beta}{\alpha}$ as t $\rightarrow \infty$; otherwise the solution grows (linearly if $a = 0$ otherwise exponentially).

The dynamics of moving to market equilibrium can be modelled by $\frac{dP}{dt} = \gamma(Q_d - Q_s)$ where $\gamma > 0$ is the *adjustment coefficient*. Then $P \uparrow$ if $Q_d > Q_s$ i.e. if demand outstrips supply the price will rise, and *vice versa*. If we have the linear supply and demand curves given above, reverting gives inverse function *Q*(*P*): explicitly
$Q_d = -\frac{P}{b} + \frac{1}{b}\ and\ Q_s = \frac{P}{d} - \frac{c}{d}$.
Substituting these gives an equation of the form $\frac{dP}{dt} = \alpha P + \beta$ and so the above theory applies.

The same equation also arises for a linear two-sector model of National Income Determination with $\frac{dY}{dt} = \gamma(C + I - Y)$ with $\gamma > 0$. By substituting for $C\ and\ I^*$ from above we get $\frac{dY}{dt} = \ \gamma(a-1)Y + \gamma(b + \ I^*)\ = \alpha Y + \beta$ so that at equilibrium (when $\frac{dY}{dt} = 0$) $Y = C + I$ when $Y = \frac{b+I^*}{1-a}$ as in the static model.

The above model is for continuous changes; for discrete changes at regular intervals, y_n, analogous equations apply. Thus if $y_n = \alpha y_{n-1} + \beta$ the solution is of the form: $y_n = A\alpha^n + \frac{\beta}{1-\alpha}$ where *A* is as above. If $a < 1$ the solution is stable and $\rightarrow -\frac{\beta}{1-\alpha}$ as $n \rightarrow \infty$.

Applications of Matrices

Suppose the market percent share of central heating fuels (electricity, gas and oil) at the start of year 0 is denoted by $\mathbf{F^0} = [F_1, F_2, F_3]$ and the following applies during each year:

- Of those using electricity, 1% will change to gas and 6% will change to oil,
- Of those using gas, 7% will change to electricity and 4% will change to oil,
- Of those using oil, 4% will change to electricity and 5% will change to gas.

The transition matrix is given by: $\mathbf{T} = \begin{pmatrix} 0.93 & 0.01 & 0.06 \\ 0.07 & 0.89 & 0.04 \\ 0.04 & 0.05 & 0.91 \end{pmatrix}$.

Then at the start of year *n*, matrix multiplication gives $\mathbf{F^n} = \mathbf{F^0}\mathbf{T}^n$. Transposing this gives $\mathbf{f^n} = \mathbf{A}^n\mathbf{f^0}$ where $\mathbf{f} = \mathbf{F}',\ \mathbf{A} = \ \mathbf{T}'$. The long-term solution may converge to a steady-state value $\mathbf{c}$ such that $\mathbf{c} = \mathbf{Ac}$ i.e. $(\mathbf{A} - \mathbf{I})\mathbf{c} = \mathbf{0}$. This is an eigenproblem (with eigenvalue 1). The solution is made unique by requiring that $c_1 + c_2 + c_3 = 100$ and substituting for c_3 gives a full-rank 2×2 matrix that can be solved.

Input/Output (I/O) analysis describes the flow of products between firms, e.g. 3 firms A, B & C that produce three quantities (available to sell) by using inputs from other firms and quite possibly, their own output too (i.e. *internal consumption*). This can be organised as an Inter-industrial flow table with outputs in columns, for example:

	A	B	C	Final demand
A input →	44	84	66	61
B input →	29	52	84	55
C input →	63	60	59	31

The table shows, e.g., that firm B provides 29 units that are used by firm A in its manufacturing process, 52 units in its own, 84 units in firm C's and sells 55 units. The *matrix of technical coefficients* is given by summing along each row (including the final demand) and then dividing elements in each column by this number. In this case:

$\mathbf{T} = \begin{pmatrix} \frac{44}{255} & \frac{84}{220} & \frac{66}{213} \\ \frac{29}{255} & \frac{52}{220} & \frac{84}{213} \\ \frac{63}{255} & \frac{60}{220} & \frac{59}{213} \end{pmatrix}$. For a given vector of output $\mathbf{x}$, the firms will produce $\mathbf{d} = \mathbf{x} - \mathbf{Tx} = (\mathbf{I} - \mathbf{T})\mathbf{x}$ and so to meet the required demand the firms must produce $\mathbf{x} = (\mathbf{I} - \mathbf{T})^{-1}\mathbf{d}$ where $(\mathbf{I} - \mathbf{T})^{-1}$ is the so-called Leontief inverse. Since these equations are linear, changes in $\mathbf{x}$ and $\mathbf{d}$ obey the same equations with $\mathbf{x}$ replaced by $\Delta\mathbf{x}$ and $\mathbf{d}$ replaced by $\Delta\mathbf{d}$.